Student Number

2024
HSC TRIAL EXAMINATION

Mathematics Advanced

General Instructions
- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using a black pen.
- Write your Student Number at the top of every page.

Total Marks – 100

Section I – Pages 3 – 5
10 marks
Allow about 15 minutes for this section.

Sections II-V – Pages 7 – 40
90 marks
- Attempt questions from questions 11 – 30.
- Answer all questions in the booklets provided.
- Allow about 2 hours and 45 minutes for this section.

This paper MUST NOT be removed from the examination room.

Section I

10 marks.
Attempt Questions 1–10.
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for questions 1 – 10.

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1. Consider the function $f(x) = |x + 2| + 1$. Which of the following is a true statement?

   (A) The function is continuous and differentiable for every value of x.
   (B) The function is not continuous but differentiable for every value of x.
   (C) The function is not continuous and not differentiable for every value of x.
   (D) The function is continuous but not differentiable for every value of x.

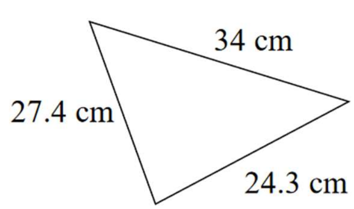
2. Consider the function $f(x) = \ln(x + \sqrt{x^2 + 1})$. Which of the following describes $f(x)$?

   (A) An even many-to-one function.
   (B) An odd many-to-one function.
   (C) An even one-to-one function.
   (D) An odd one-to-one function.

3. For what values of k does the quadratic equation $x^2 + kx - 2k = 0$ have real roots?

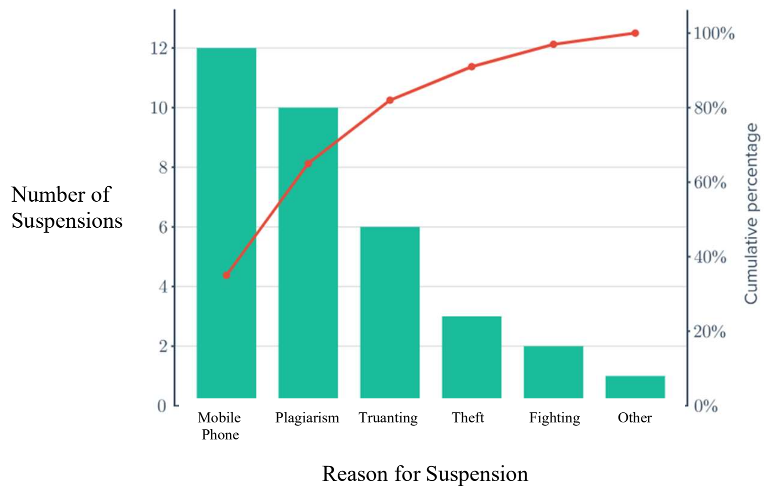
   (A) $k \in (-\infty, -8) \cup (0, \infty)$
   (B) $k \in (-\infty, -8] \cup [0, \infty)$
   (C) $k \in (-8,0)$
   (D) $k \in [-8,0]$

4. Which of the following gives the magnitude of the smallest angle in the triangle below to the nearest degree?



   (A) 30°
   (B) 45°
   (C) 53°
   (D) 79°

5. The following pareto chart shows the reasons for suspensions at a school.



Which of the following accounted for more than 80% of suspensions?

(A) Plagiarism.
(B) Mobile Phone.
(C) Mobile Phone, Plagiarism and Truanting.
(D) Truanting, Theft, Fighting and Other.

6. A and B are events such that:

$$P(A \cap B) = \frac{2}{5} \quad \text{and} \quad P(A \cap B') = \frac{3}{7}$$

Which of the following is the value of $P(B'|A)$?

(A) $\frac{6}{35}$
(B) $\frac{15}{29}$
(C) $\frac{14}{35}$
(D) $\frac{29}{35}$

7. Consider the discrete probability distribution function with random variable X defined in the table below:

x	-2	0	b	$2b$	4
$P(X = x)$	a	b	b	$2b$	0.2

Which of the following is true for the expected value $E(X)$?

(A) $-0.8 \leq E(X) \leq 1$
(B) $0 \leq E(X) \leq 1$
(C) $0.2 \leq E(X) \leq 0.8$
(D) $0 \leq E(X) \leq 2.4$

8. Which of the following is the value of the gradient of the tangent to $y = -\frac{1}{x}$ at the point $\left(2, \frac{-1}{2}\right)$?

(A) 4
(B) -4
(C) $\frac{1}{4}$
(D) $-\frac{1}{4}$

9. The graph of a function f is obtained when the graph of function g with the rule $g(x) = x^2 - 3x - 10$ is translated by 7 units in the vertical direction followed by a reflection across the y-axis.

Which of the following is the rule for the function f?

(A) $x^2 + 3x - 3$
(B) $-x^2 + 3x + 3$
(C) $x^2 - 2x - 22$
(D) $-x^2 + 2x + 22$

10. If $\int_2^6 f(x)dx = 8$, to which of the following is $\int_4^8 2f(x-2)dx$ equal?

(A) 4
(B) 6
(C) 16
(D) None of the above.

Student Number

Question 11 (4 marks) **Marks**

Katie has deliberately designed a biased six-sided die with the following probability distribution for X, the number showing on the uppermost face when the die is rolled.

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1-\theta}{6}$	$\frac{1-\theta}{6}$	$\frac{1-\theta}{6}$	$\frac{1+\theta}{6}$	$\frac{1+\theta}{6}$	$\frac{1+\theta}{6}$

(a) What values of θ allow for $P(X)$ to be a probability distribution function? **2**

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(b) Find $P(1 \leq X \leq 4)$ in terms of θ. **1**

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(c) Find the probability of rolling an even number in terms of θ. **1**

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Student Number

Question 12 (5 marks) **Marks**

Janet and Alan are playing a tennis match. The probability of Janet winning the first set is 0.3. After that, Janet's probability of winning a set is 0.6 if she won the previous set, or 0.4 if she lost the previous set. The match will continue until either Janet or Alan wins two sets.

(a) Draw a probability tree diagram to describe the match. **2**

(b) Find the expected value of the number of sets that the match will last. **2**

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(c) Find the probability that Alan wins, given the match lasted 3 sets. **1**

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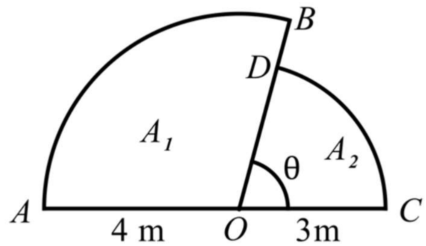
Student Number

Question 13 (2 marks) **Marks**

Two sectors A_1 and A_2 are constructed along the interval AC. $AO = 4$ and $OC = 3$ as shown in the diagram below.



If the area of A_1 is twice the area of A_2, find the value of θ. **2**

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Question 14 (2 marks)

Prove: $\frac{\cot x + \operatorname{cosec} x}{\tan x + \sin x} = \cot x \operatorname{cosec} x$ **2**

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Student Number

Question 15 (5 marks) **Marks**

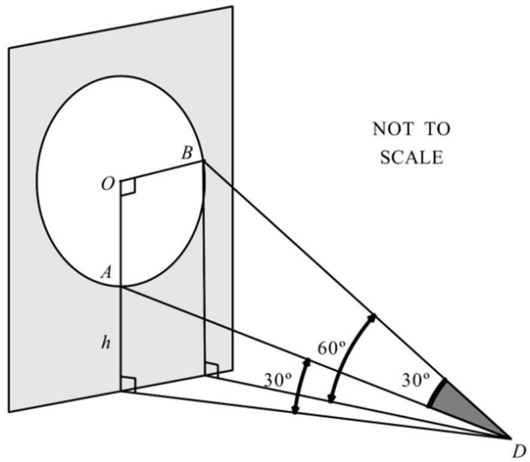
Solve $2\cos^2 x - 3\cos x = 2$ for $0 \leq x \leq 2\pi$, leaving your answer in exact form. **3**

(b) Solve $3\sin x + \sqrt{3}\cos x = 0$ over the domain $0 \leq x \leq 2\pi$. **2**

Student Number

Question 16 (5 marks) **Marks**

A circle with centre O and radius 10m is drawn on a vertical wall, standing on a horizontal floor. Point A is vertically below O and $\angle AOB = 90°$ as shown in the diagram below.



A laser pointer is aimed from a point D on the ground. The angles of elevation of A and B from D are 30° and 60° respectively. Let A be h metres above the ground and $\angle ADB = 30°$.

(a) Write expressions for AD and BD in terms of h. **2**

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(b) Hence find the exact value of h. **3**

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Student Number

Mathematics Advanced
Section III Answer Booklet

21 marks.
Attempt Questions 17–20.

Instructions

- Write you student number at the top of each page.
- Answer the questions in the spaces provided. These spaces provide guidance for the expected length of response.
- Your response should include relevant mathematical reasoning and/or calculations.
- Extra writing space is provided at the back of this booklet. If you use this space, clearly indicate which question you are answering.

Student Number

Question 17 (7 marks) **Marks**

Consider the curve $y = x^3 - 6x^2 + 7$.

(a) Find the coordinates of any stationary points and determine their nature. **3**

(b) Find the coordinates of any points of inflexion. **2**

(c) Sketch the graph, clearly showing all the points found in (a) and (b) and y-intercepts. You do NOT need to find x-intercepts. **2**

Student Number

Question 18 (5 marks) **Marks**

(a) Differentiate xe^{5x} with respect to x. **2**

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(b) Hence evaluate $\int_0^{10} 5xe^{5x}dx$ **3**

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Student Number

Question 19 (3 marks) **Marks**

Differentiate $f(x) = x^2 - 4x + 5$ from first principles. **3**

Student Number

Question 20 (6 marks) **Marks**

The velocity of a particle travelling on the x axis is given by $v(t) = \frac{1}{t+1} - \frac{1}{2}$ for $t \geq 0$.

(a) Show that the particle is stationary when $t = 1$. **1**

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(b) Explain why the particle is moving to the left for all times $t > 1$. **2**

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(c) Find the distance travelled by the particle in the first 3 seconds, correct to one decimal place. **3**

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Student Number

Mathematics Advanced
Section IV Answer Booklet

23 marks.
Attempt Questions 21–25.

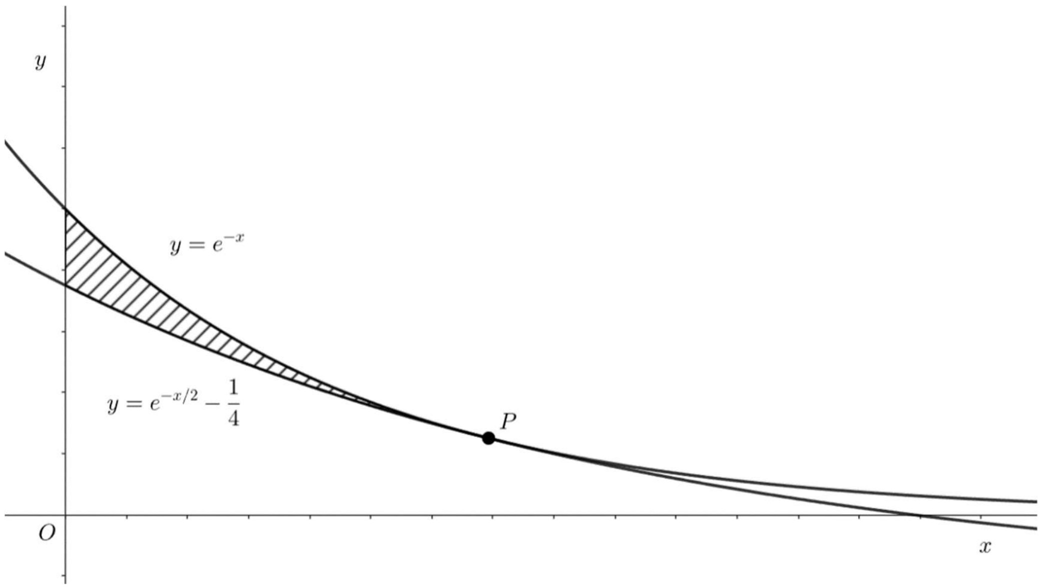
Instructions

- Write you student number at the top of each page.
- Answer the questions in the spaces provided. These spaces provide guidance for the expected length of response.
- Your response should include relevant mathematical reasoning and/or calculations.
- Extra writing space is provided at the back of this booklet. If you use this space, clearly indicate which question you are answering.

Student Number

Question 21 (3 marks) **Marks**

The curves $y = e^{-x}$ and $y = e^{-\frac{x}{2}} - \frac{1}{4}$ intersect at the point P as shown in the diagram below.



(a) Show that the point of intersection, P, is $\left(\ln 4, \frac{1}{4}\right)$. **1**

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(b) Hence find the area bounded by the two curves and the y-axis, as shaded in the diagram. **2**

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Student Number

Question 22 (2 marks) **Marks**

Find: **2**

$$\int \frac{1-x}{x^3}\,dx$$

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Student Number

Question 23 (3 marks) **Marks**

A discrete random variable has a probability distribution as shown in the table below where n is a finite, positive integer and r is any real number other than 1.

x	r	r^2	r^3	$\cdots$	r^k	$\cdots$	r^n
$P(X = x)$	r^n	r^{n-1}	r^{n-2}	$\cdots$	r^{n-k+1}	$\cdots$	r

(a) Show that $E(X) = n(2r - 1)$. **3**

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Student Number

Question 24 (8 marks) **Marks**

Consider $f(x) = \ln[\cos(3x)]$ where $-\frac{\pi}{6} < x < \frac{\pi}{6}\cdot$

(a) Find $f'(x)$ in simplest form. **1**

(b) Show that the equation of the tangent to $y = \tan 3x$ at $x = \frac{\pi}{12}$ is given by **3**

$y = 6x + 1 - \frac{\pi}{2}$.

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(c) Find the exact value of the x intercept of the tangent found in (b). **1**

(d) Find the area bound by the curve $y = \tan 3x$, the tangent and the x axis. **3**

Give your answer correct to four decimal places.

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Question 25 (7 marks) **Marks**

The depth of the water in a harbour changes due to the tides and can be modelled with the equation $d = 12 + 4\cos\left(\frac{t}{3} - \frac{\pi}{6}\right)$, where d is the depth of the harbour in metres and t is the number of hours since midnight on Tuesday morning.

(a) Draw a neat sketch of the depth of the water d for $t \in [0, 6\pi]$. **3**

(b) A boat needs 10 metres of water to be operated in the harbour, although if it motors out of the harbour, the water is always deep enough to operate. The boat uses 6 litres of petrol every hour that it is being used. If the boat is launched at 7:30 AM and leaves the harbour, and it takes 12 minutes to motor each way in and out of the harbour, what is the minimum amount of petrol that the boat must hold to make it back to its starting point? Answer to the nearest litre. **4**

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Student Number

Mathematics Advanced
Section V Answer Booklet

23 marks.
Attempt Questions 26–30.

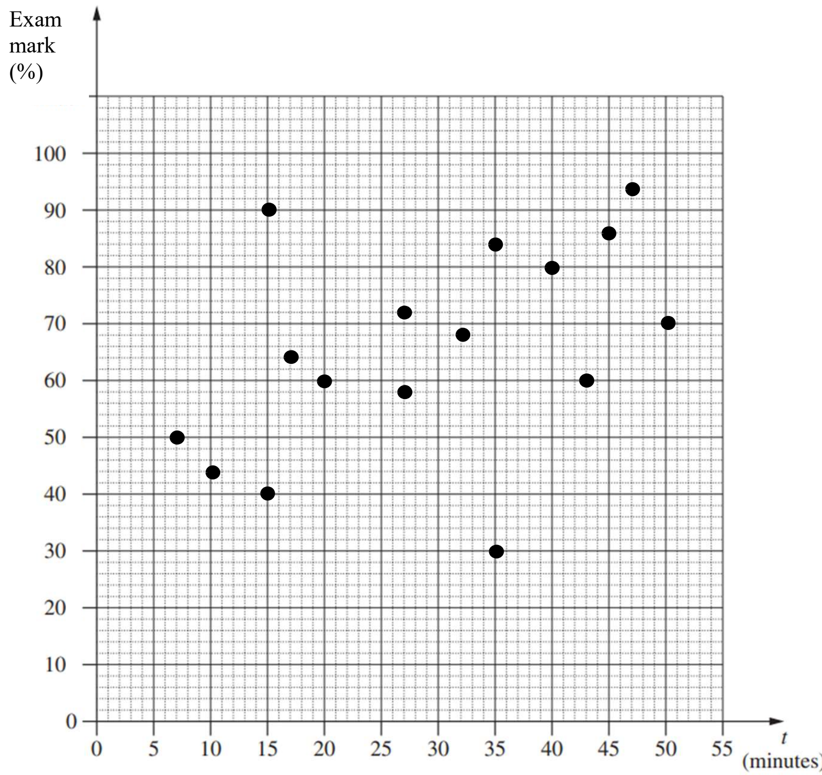
Instructions

- Write you student number at the top of each page.
- Answer the questions in the spaces provided. These spaces provide guidance for the expected length of response.
- Your response should include relevant mathematical reasoning and/or calculations.
- Extra writing space is provided at the back of this booklet. If you use this space, clearly indicate which question you are answering.

Student Number

Question 26 (5 marks) **Marks**

A researcher conducted a survey of 16 students at a school, asking them for the number of minutes they spent on their device each night and the mark they received in their last examination as a percentage. He plotted the data on the graph below:



(a) Calculate the least squares regression line for the data. **1**

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(b) With evidence, describe the strength of the correlation between minutes spent on a device by a student and their mark in an examination. **1**

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(c) A new student is surveyed who spends 20 minutes on their device each night. **1**

Predict their score on the next examination to the nearest percentage.

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(d) Another new student wanted to use the model in (a) to predict their next examination **2**
but was told that their time spent on their device was too large. What is the smallest amount of time the student could have reported spending on their device? Assume the graduation in time measurement is one minute.

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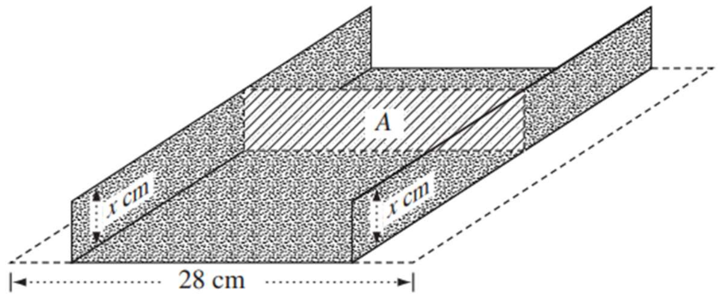
Student Number

Question 27 (5 marks) **Marks**

A long rectangular sheet of metal 28cm wide is to be made into a gutter by turning up sides of equal height $x$cm, perpendicular to the base as shown below:



(a) Show that the formula for the cross-sectional area, A, of the gutter is **1**

$$A = 28x - 2x^2$$

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(b) Explain why this formula is only valid for $x \in (0,14)$. **1**

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(c) Find the maximum value of A. **3**

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Student Number

Question 28 (4 marks) **Marks**

A new artist releases a song on a music streaming platform. The number of 'listens' each hour for the first 5 hours is recorded in the table below.

Hour (H)	1	2	3	4	5
Listens (L)	13	36	62	94	138

(a) Show that the predicted number of listens in the 6^{th} hour is 206. **2**

(b) How many predicted 'listens' will the song have had at the conclusion of one day? **2**

Student Number

Question 29 (5 marks) **Marks**

A toxic substance in a lake is degrading naturally according to the formula $A(t) = 3000m^{-0.2t}$, where A is the number of grams of the substance remaining at t years and $m > 0$ is a constant.

(a) If $A = 2400$ after 6 years, find the value of m correct to 1 decimal place. **1**

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(b) Fish can be reintroduced to the lake once the rate of change of the toxic substance is greater than -20. After how many years can fish be reintroduced? Answer to one decimal place. **3**

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(c) People can swim in the lake once the amount of the toxic substance is less than 50. In how many years can this happen, to the nearest year? **1**

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Student Number

Question 30 (4 marks) **Marks**

A continuous random variable, X, has the following probability density function:

$$f(x) = \begin{cases} \dfrac{3\sqrt{x}}{2(\ln 2)\sqrt{x^3}} & \text{for } k \le x \le 8 \\ 0 & \text{for all other values of } x \text{ in } \mathbb{R} \end{cases}$$

(a) Find the value of k. Leave your answer as an exact value. **2**

(b) Find the median value of X as an exact value. **2**

End of paper.

MC
1) D 3) B 5) C 7) A 9) A
2) D 4) B 6) B 8) C 10) C

Student Number

Question 11 (4 marks)

Marks

Katie has deliberately designed a biased six-sided die with the following probability distribution for X, the number showing on the uppermost face when the die is rolled.

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1-\theta}{6}$	$\frac{1-\theta}{6}$	$\frac{1-\theta}{6}$	$\frac{1+\theta}{6}$	$\frac{1+\theta}{6}$	$\frac{1+\theta}{6}$

(a) What values of θ allow for $P(X)$ to be a probability distribution function? **2**

* must show

$\leq P(X = x) = 1$

$1 = \frac{1-\theta}{6} + \frac{1-\theta}{6} + \frac{1-\theta}{6} + \frac{1+\theta}{6} + \frac{1+\theta}{6} + \frac{1+\theta}{6}$

$= \frac{3(1-\theta)}{6} + \frac{3(1+\theta)}{6}$

(no solution

Many students wrote $(-1 < \theta < 1)$ ✗

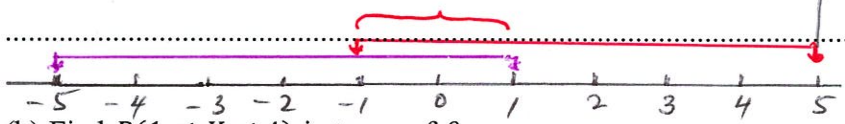
$0 \leq \frac{1-\theta}{6} \leq 1$

$1 \geq \theta \geq -5$

or

$0 \leq \frac{1+\theta}{6} \leq 1$

$-1 \leq \theta \leq 5$ — (1)



$\therefore -1 \leq \theta \leq 1$ — (1)

(b) Find $P(1 \leq X \leq 4)$ in terms of θ. **1**

$\frac{1-\theta}{6} + \frac{1-\theta}{6} + \frac{1-\theta}{6} + \frac{1+\theta}{6}$

$= \frac{4-2\theta}{6} = \frac{2-\theta}{3}$ — (1)

(c) Find the probability of rolling an even number in terms of θ. **1**

$P(x \text{ even}) = \frac{1-\theta}{6} + \frac{1+\theta}{6} + \frac{1+\theta}{6}$

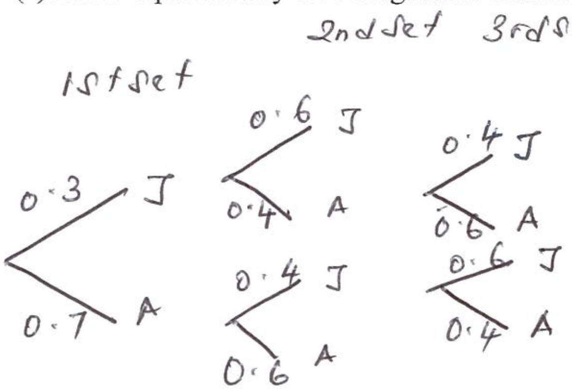
$= \frac{3+\theta}{6}$ — (1)

Question 12 (5 marks) **Marks**

Student Number

Janet and Alan are playing a tennis match. The probability of Janet winning the first set is 0.3. After that, Janet's probability of winning a set is 0.6 if she won the previous set, or 0.4 if she lost the previous set. The match will continue until either Janet or Alan wins two sets.

(a) Draw a probability tree diagram to describe the match. **2**



JJ = 0.18
JAJ = 0.048
JAA = 0.072
AJJ = 0.168
AJA = 0.112
AA = 0.42

(1) — tree diagram correct to 2nd set

(1) — tree diagram correct to 3rd set

(b) Find the expected value of the number of sets that the match will last. **2**

x	2	3
$P(X=x)$	3/5	2/5
	0.6	0.4

— (1)

$E(X) = 2 \times 0.6 + 3 \times 0.4$

$= 2.4$ — (1)

(c) Find the probability that Alan wins, given the match lasted 3 sets. **1**

$P(\text{A wins} \mid 3 \text{ sets}) = \dfrac{0.072 + 0.112}{0.048 + 0.072 + 0.168 + 0.112}$

$= 0.46$ — (1)

Question 13 (2 marks) **Marks**

Two sectors A_1 and A_2 are constructed along the interval AC. $AO = 4$ and $OC = 3$ as shown in the diagram below.

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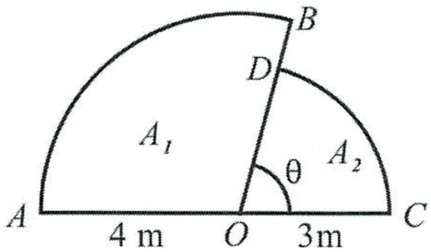


If the area of A_1 is twice the area of A_2, find the value of θ. **2**

Formula in reference sheet
no cfe for wrong formula

$A_1 = 2 \times A_2$

$\frac{1}{2}(4)^2(\pi - \theta) = 2 \times \frac{1}{2}(3)^2\theta$ — (1)

$\frac{16}{2}(\pi - \theta) = 9\theta$

$17\theta = 8\pi$

$\theta = \frac{8\pi}{17}$ or $84°42'$ or $84.71°$ — (1)

Question 14 (2 marks)

Prove: $\frac{\cot x + \operatorname{cosec} x}{\tan x + \sin x} = \cot x \operatorname{cosec} x$ **2**

M1

$LHS = \left(\frac{\cos x}{\sin x} + \frac{1}{\sin x}\right) \div \left(\frac{\sin x}{\cos x} + \frac{\sin x}{1}\right)$

$= \frac{\cos x(\cos x + 1)}{\sin^2 x + \sin^2 x \cos x}$

$= \frac{\cos x(1 + \cos x)}{\sin^2 x(1 + \cos x)}$ — (1)

(1) — $= \frac{\cos x}{\sin x} \times \frac{1}{\sin x} = \cot x \operatorname{cosec} x$

$= RHS$

M2

$LHS = \left(\frac{1}{\tan x} + \frac{1}{\sin x}\right) \times \frac{1}{\tan x + \sin x}$

(1) — $= \frac{\sin x + \tan x}{\tan x \sin x} \times \frac{1}{\tan x + \sin x}$

$= \frac{1}{\tan x} \times \frac{1}{\sin x}$ — (1)

$= \cot x \operatorname{cosec} x$

$= RHS$

Question 15 (5 marks) **Marks**

Solve $2\cos^2 x - 3\cos x = 2$ for $0 \leq x \leq 2\pi$, leaving your answer in exact form. **3**

Student Number

$2\cos^2 x - 3\cos x - 2 = 0 \quad (0 \le x \le 2\pi)$

$2\cos x \quad +1 \quad \cos x$

$\cos x \quad -2 \quad 4\cos x$

$(2\cos x + 1)(\cos x - 2) = 0$

$\cos x = -\frac{1}{2}$ or $\cos x = 2$ (no solution) — ①

✓ $x = \pi/3$

✓ $\therefore x = \frac{2\pi}{3}, \frac{4\pi}{3}$ ②

(b) Solve $3\sin x + \sqrt{3}\cos x = 0$ over the domain $0 \le x \le 2\pi$. **2**

$3\sin x = -\sqrt{3}\cos x$ $(\div \cos x)$

$3\tan x = -\sqrt{3}$

$x = \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

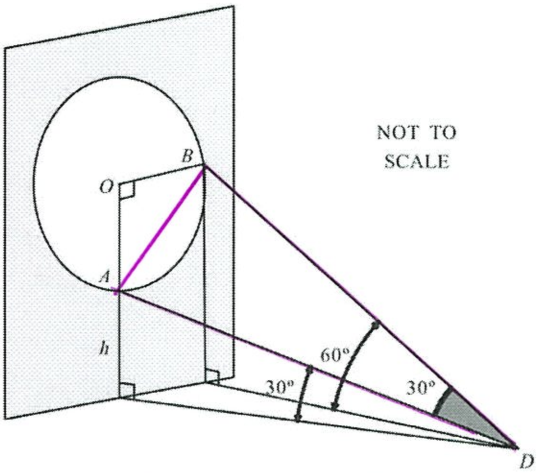
acute $x = \pi/6$

✓ $\Rightarrow x = \frac{5\pi}{6}, \frac{11\pi}{6}$ ✓

Question 16 (5 marks) **Marks**

A circle with centre O and radius 10m is drawn on a vertical wall, standing on a horizontal floor. Point A is vertically below O and $\angle AOB = 90°$ as shown in the diagram below.

Student Number



A laser pointer is aimed from a point D on the ground. The angles of elevation of A and B from D are $30°$ and $60°$ respectively. Let A be h metres above the ground and $\angle ADB = 30°$.

(a) Write expressions for AD and BD in terms of h. **2**

$\frac{h}{\sin 30} = AD$ $\quad BD = \frac{h+10}{\sin 60}$

$AD = 2h$ — ① $\quad BD = \frac{2(h+10)}{\sqrt{3}}$ — ①

(b) Hence find the exact value of h. **3**

In ΔOAB, $AB = 10\sqrt{2}$ (Pythagoras)

In ΔABD (cos rule) ①

$(10\sqrt{2})^2 = \left(\frac{2(h+10)}{\sqrt{3}}\right)^2 + (2h)^2 - 2\left[\left(\frac{2(h+10)}{\sqrt{3}}\right)(2h)\right]\cos 30°$

$200 = \frac{4h^2 + 80h + 400}{3} + 4h^2 - 2(2h)\frac{2(h+10)}{\sqrt{3}} \times \frac{\sqrt{3}}{2}$

$\left(200 = \frac{4h^2 + 80h + 400}{3} - 4h^2 - 4h^2 - 40h\right) \times 3$

$600 = 4h^2 + 80h + 400 - 120h$

$0 = 4h^2 - 40h - 200 \rightarrow 0 = 4(h^2 - 10h - 50)$

Section II extra writing space.

$\therefore h = 5 \pm 5\sqrt{3}$ $\quad h > 0$

If you use this space, clearly indicate which question(s) you are answering. Leave at least two lines between each question.

$\therefore h = 5 + 5\sqrt{3}$ or 13.66 ①

Student Number

Question 17 (7 marks) **Marks**

Consider the curve $y = x^3 - 6x^2 + 7$.

(a) Find the coordinates of any stationary points and determine their nature. **3**

For stat pts need $y' = 0$

i.e. $3x^2 - 12x = 0$

$3x(x-4) = 0$ ✓

$x = 0$ or $x = 4$ ✓ 2 marks for finding both x-values.

$y = 7$ $y = -25$

$y'' = 6x - 12$

at $(0,7)$: $y'' = -12 < 0$ ∴ maximum at $(0,7)$

at $(4,-25)$: $y'' = 12 > 0$ ∴ minimum at $(4,-25)$ ✓

3rd mark for appropriate method determining nature + finding y-co-ordinates

(b) Find the coordinates of any points of inflexion. **2**

For point of inflection need $y'' = 0$

i.e. $6x - 12 = 0$

$x = 2$ ✓ $y = -9$ — First mark finding x-value

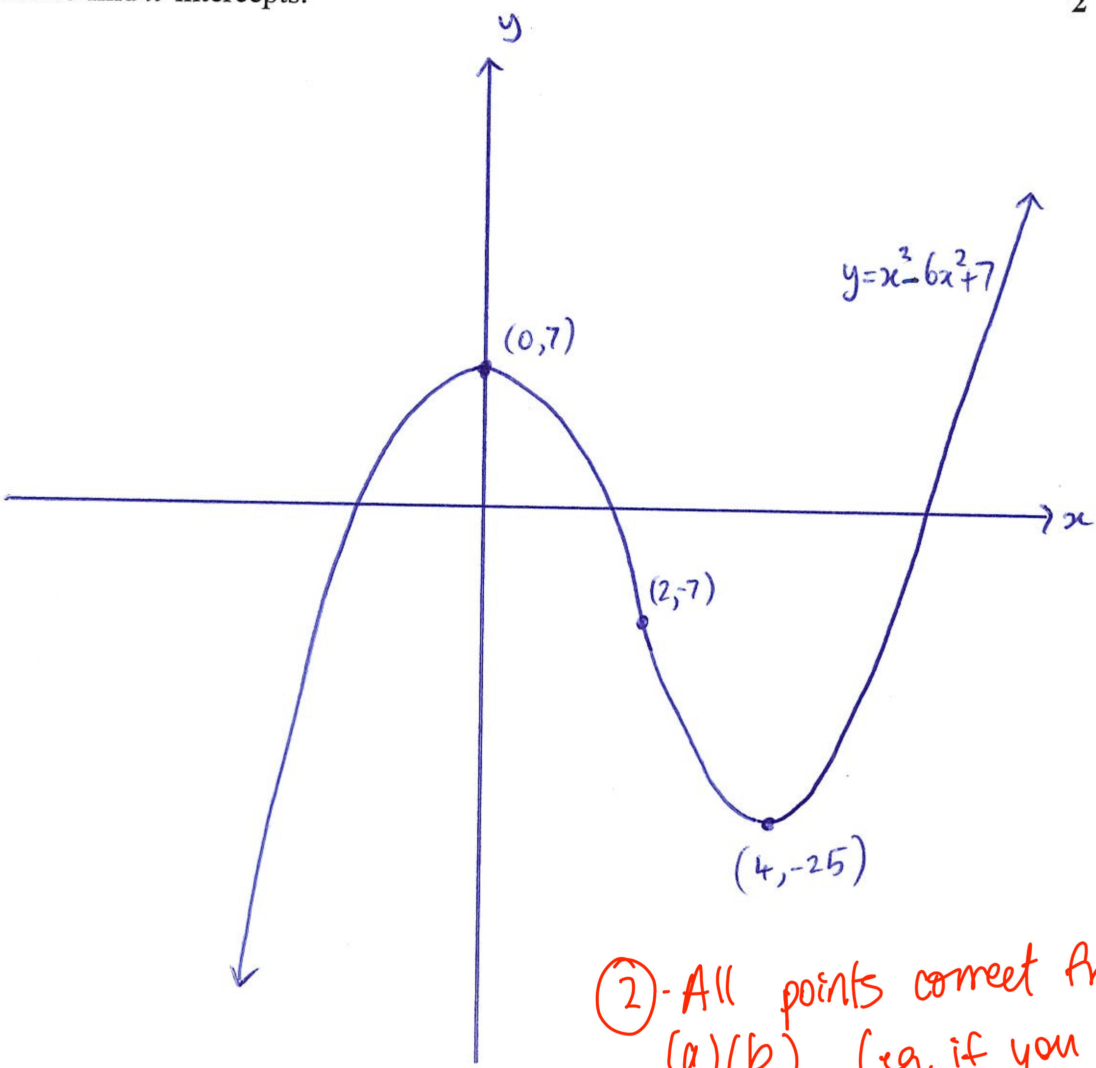
x	1.9	2	2.1
y''	-0.6	0	0.6
	∩		∪

∴ change in concavity at $(2,-9)$

∴ point of inflection at $(2,-9)$ ✓

- second mark for testing concavity change

(c) Sketch the graph, clearly showing all the points found in (a) and (b) and y-intercepts. You do NOT need to find x-intercepts. **2**



(2) - All points correct from (a)(b) (eg. if you found a maximum, must draw maximum)
- relatively to scale
most scale issues due to y-axis.

(1/2) All points in (a),(b) marked in + general cubic polynomial shape.

Student Number

Question 18 (5 marks) **Marks**

(a) Differentiate xe^{5x} with respect to x. **2**

$\frac{d}{dx} xe^{5x} = x.5e^{5x} + e^{5x}$ ✓✓

$= e^{5x}(5x+1)$

Mostly done well.

(b) Hence evaluate $\int_0^{10} 5xe^{5x}dx$ **3**

$\int_0^{10} 5xe^{5x}dx = \int_0^{10} 5xe^{5x} + e^{5x} - e^{5x}\, dx$ ✓ 1st mark: use hence

$= \left[xe^{5x} - \frac{e^{5x}}{5}\right]_0^{10}$ from (a)

✓ 2nd mark: integrate correctly

$= 10e^{50} - \frac{e^{50}}{5} - (0 - \frac{1}{5})$

$= \frac{49e^{50}+1}{5}$ ✓ 3rd mark correct substitution, not necessarily fully simplified.

$\frac{2}{3}$ given if 1-2 arithmetic errors made.

Student Number

Question 19 (3 marks) **Marks**

Differentiate $f(x) = x^2 - 4x + 5$ from first principles. **3**

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - 4(x+h) + 5 - x^2 + 4x - 5}{h}$$

$$= \lim_{h \to 0} \frac{(x+h-x)(x+h+x) - 4x - 4h + 4x}{h}$$ ✓ substitution

$$= \lim_{h \to 0} \frac{h(2x+h) - 4h}{h}$$ ✓ Expand and simplify

$$= \lim_{h \to 0} \frac{h(2x+h-4)}{h}$$

$$= 2x + 0 - 4$$ ✓ correct cancellation of h & substitution of $h \to 0$.

$$\therefore f'(x) = 2x - 4$$

* Must use first principles for any marks.

Student Number

Question 20 (6 marks) **Marks**

The velocity of a particle travelling on the x axis is given by $v(t) = \frac{1}{t+1} - \frac{1}{2}$ for $t \geq 0$.

(a) Show that the particle is stationary when $t = 1$. **1**

$v(1) = \frac{1}{1+1} - \frac{1}{2}$

$= 0 \quad \therefore$ stationary at $t=1$

Must show substitution AND concluding statement

(b) Explain why the particle is moving to the left for all times $t > 1$. **2**

$a(t) = v'(t)$

$= \frac{-1}{(t+1)^2}$

$\therefore a(t) < 0 \quad \forall t \geq 0$

Since $v(1) = 0$ and $a(1) < 0$ the particle will move to the left after $t=1$. Since $a(t) < 0 \ \forall t > 1$ the particle will always move to the left for $t > 1$.

✓ Mathematical reasoning eg. Finding $a(t)$ or use of inequalities

✓ Using words to justify why.

(c) Find the distance travelled by the particle in the first 3 seconds, correct to one decimal place. **3**

$d = \int_0^1 \frac{1}{t+1} - \frac{1}{2}\,dt + \left|\int_1^3 \frac{1}{t+1} - \frac{1}{2}\,dt\right|$ ✓ set up

$= \left[\ln|t+1| - \frac{t}{2}\right]_0^1 + \left|\left[\ln|t+1| - \frac{t}{2}\right]_1^3\right|$ ✓ correct integration

$= \ln 2 - \frac{1}{2} - (0 - 0) + \left|\ln 4 - \frac{3}{2} - \ln 2 + \frac{1}{2}\right|$

$= \ln 2 - \frac{1}{2} + \ln 2 - \ln 4 + 1$

$= \ln \frac{2 \times 2}{4} + \frac{1}{2}$

$= \frac{1}{2}$

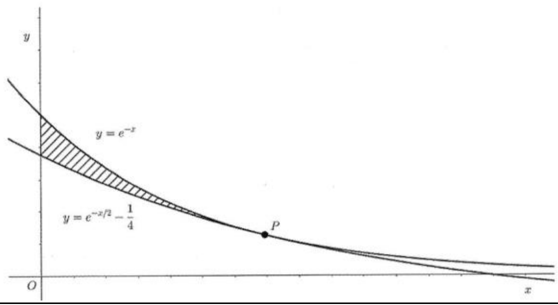
$\therefore d = 0.5$ ✓

Final mark for correct substitution. No marks awarded if integrated expression has no logs.

Q21

Question 21 (3 marks)

The curves $y = e^{-x}$ and $y = e^{-\frac{x}{2}} - \frac{1}{4}$ intersect at the point P as shown in the diagram below.



(a) Show that the point of intersection, P, is $\left(\ln 4, \frac{1}{4}\right)$. **1**

$e^{-x} = e^{-\frac{x}{2}} - \frac{1}{4}$

$4 = 4e^{\frac{x}{2}} - e^{x}$

$e^{x} - 4e^{\frac{x}{2}} + 4 = 0$

$(e^{\frac{x}{2}} - 2)^2 = 0$

$e^{\frac{x}{2}} = 2$

$\frac{x}{2} = \ln 2$

$x = \ln 4$

sub into $y = e^{-x}$

$y = e^{-\ln 4}$

$y = e^{\ln \frac{1}{4}}$

$y = \frac{1}{4}$

$\therefore P(\ln 4, \frac{1}{4})$

① Substitution method is allowed as long as clear workingouts is shown.

- Students will lose a mark if their working is not coherent.
- Common mistakes:
  e.g. $\frac{1}{4} = e^{-x} - e^{-\frac{x}{2}}$
  $\ln \frac{1}{4} = \ln e^{-x} - \ln e^{-\frac{x}{2}}$

(b) Hence find the area bounded by the two curves and the y-axis, as shaded in the diagram. **2**

$A = \int_0^{\ln 4} e^{-x} - e^{-\frac{x}{2}} + \frac{1}{4} \, dx$ ①

$= \left[-e^{-x} + 2e^{-\frac{x}{2}} + \frac{x}{4} \right]_0^{\ln 4}$

$= -e^{-\ln 4} + 2e^{-\frac{\ln 4}{2}} + \frac{\ln 4}{4} - (-1 + 2 + 0)$

$= -\frac{1}{4} + 2 \cdot \frac{1}{2} + \frac{\ln 4}{4} - 1$

$\therefore A = \frac{\ln 4 - 1}{4}$ units^2 ①

- 1 mark for showing the correct setting up for the integration } Generally well done

- 1 mark for final sol; different forms are accepted.
  e.g. $\frac{2\ln 2 - 1}{4}$, ≈ 0.097

Q22

$$\int \frac{1-x}{x^3} dx$$

$\int \frac{1-x}{x^3} dx = \int x^{-3} - x^{-2} dx$ ①

$= -\frac{x^{-2}}{2} + x^{-1} + C$

$= \frac{1}{x} - \frac{1}{2x^2} + C$ ①

Well done.

- Some used the quotient rule.

- $\frac{2x-1}{2x^2} + C$ ✓

* -1 for missing '+C', for the entire paper.

Q23.

Question 23 (3 marks) **Marks**

A discrete random variable has a probability distribution as shown in the table below where n is a finite, positive integer and r is any real number other than 1.

x	r	r^2	r^3	...	r^k	...	r^n
$P(X = x)$	r^n	r^{n-1}	r^{n-2}	...	r^{n-k+1}	...	r

(a) Show that $E(X) = n(2r - 1)$. **3**

$E(X) = r.r^n + r^2.r^{n-1} + r^3.r^{n-2} + \dots + r^k.r^{n-k+1} + \dots + r^n.r$

$= r^{n+1} + r^{n+1} + r^{n+1} + \dots + r^{n+1} + \dots + r^{n+1}$

$= nr^{n+1}$ (1) ① Correct expression for E(X)

most students got this

many forgot about this

Also, $r^n + r^{n-1} + r^{n-2} + \dots + r = 1$ (probabilities of pdf)

$\frac{r(r^n - 1)}{r-1} = 1$ ① $\sum P(X=x) = 1$ w/ some working out. (e.g. recognising GP)

$r(r^n - 1) = r - 1$

$r^{n+1} - r = r - 1$

$r^{n+1} = 2r - 1$

sub into (1):

$E(X) = n(2r-1)$ as required. ① Substitution

Question 24 (8 marks) **Marks**

Consider $f(x) = \ln[\cos(3x)]$ where $-\frac{\pi}{6} < x < \frac{\pi}{6}$.

Generally well done.

(a) Find $f'(x)$ in the simplest form. **1**

$f'(x) = \frac{-3\sin 3x}{\cos 3x}$

$= -3\tan 3x$

(b) Show that the equation of the tangent to $y = \tan 3x$ at $x = \frac{\pi}{12}$ is given by **3**

$y = 6x + 1 - \frac{\pi}{2}$.

when $x = \frac{\pi}{12}$, $y = \tan \frac{3\pi}{12}$

$= \tan \frac{\pi}{4}$

$= 1$ $\therefore \left(\frac{\pi}{12}, 1\right)$

$y' = 3\sec^2 3x$ (1) Correct $\frac{dy}{dx}$

At $x = \frac{\pi}{12}$: $m_T = 3\sec^2 \frac{\pi}{4}$

$= 3(\sqrt{2})^2$

$= 6$ (1) correct $m = 6$

$\therefore$ eqn tangent:

$y - 1 = 6\left(x - \frac{\pi}{12}\right)$

$y - 1 = 6x - \frac{\pi}{2}$ } Show

$y = 6x + 1 - \frac{\pi}{2}$

$(y - y_1) = m(x - x_1)$

OR $y = mx + b$.

Well done.

(c) Find the exact value of the x intercept of the tangent found in (b). **1**

$y = 0 \Rightarrow 6x + 1 - \frac{\pi}{2} = 0$

$x = \frac{\pi}{12} - \frac{1}{6}$ (1)

$\frac{\frac{\pi}{2} - 1}{6}$ is accepted but simplified version is preferred.

(d) Find the area bound by the curve $y = \tan 3x$, the tangent and the x axis. **3**

Give your answer correct to four decimal places.

$$A = \int_0^{\frac{\pi}{12}} \tan 3x \, dx - \frac{1}{2}\left(\frac{\pi}{12} - \left(\frac{\pi}{12} - \frac{1}{6}\right)\right)(1) \quad ①$$

$$= -\frac{1}{3}\int_0^{\frac{\pi}{12}} -3\tan 3x \, dx - \frac{1}{2}\left(\frac{1}{6}\right)(1)$$

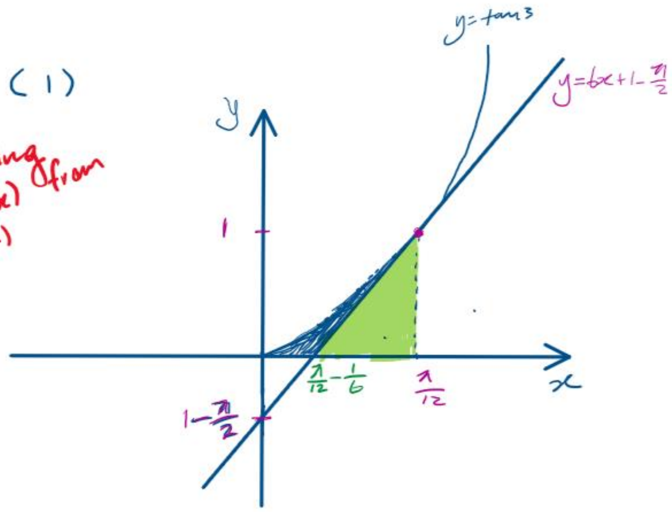
$$= -\frac{1}{3}\left[\ln \cos(3x)\right]_0^{\frac{\pi}{12}} - \frac{1}{12}$$

① showing $\ln \cos(3x)$ from part a)

$$= -\frac{1}{3}\ln\sqrt{2} - \frac{1}{12}$$

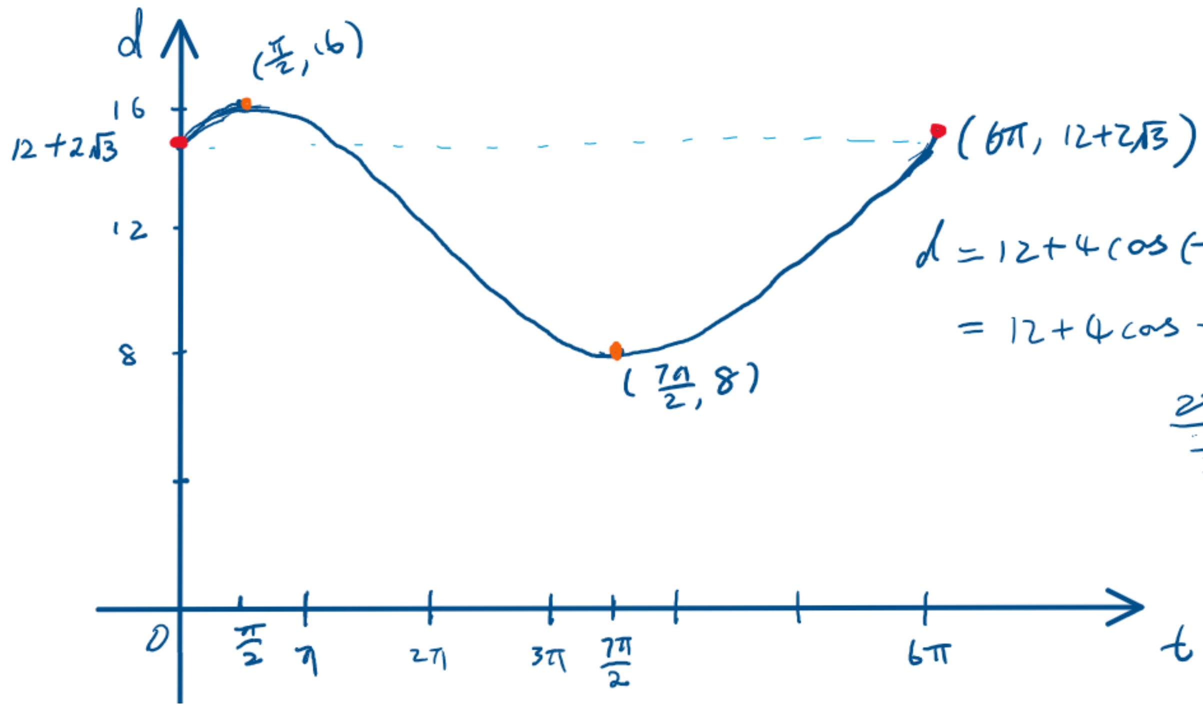
$$= 0.0321912 \ldots$$

$$= 0.0322 \text{ (3.d.p).} \quad ①$$



Q25

The depth of the water in a harbour changes due to the tides and can be modelled with the equation $d = 12 + 4\cos\left(\frac{t}{3} - \frac{\pi}{6}\right)$, where d is the depth of the harbour in metres and t is the number of hours since midnight on Tuesday morning.



$d = 12 + 4\cos\left(\frac{t}{3} - \frac{\pi}{6}\right)$

$= 12 + 4\cos\frac{1}{3}\left(t - \frac{\pi}{2}\right)$

$\frac{2\pi}{\frac{1}{3}} = 6\pi$

① Correct min & max (both coordinates must be clearly labelled)

① Correct shape (a common mistake: e.g.

)

① Both end points

(b) A boat needs 10 metres of water to be operated in the harbour, although if it motors out of the harbour, the water is always deep enough to operate. The boat uses 6 litres of petrol every hour that it is being used. If the boat is launched at 7:30 AM and leaves the harbour, and it takes 12 minutes to motor each way in and out of the harbour, what is the minimum amount of petrol that the boat must hold to make it back to its starting point? Answer to the nearest litre. **4**

$10 = 12 + 4\cos\left(\frac{t}{3} - \frac{\pi}{6}\right)$

$-\frac{1}{2} = \cos\left(\frac{t}{3} - \frac{\pi}{6}\right)$ (1)

$\frac{t}{3} - \frac{\pi}{6} = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \ldots$ * Graphical method

$t - \frac{\pi}{2} = 2\pi, 4\pi, 8\pi, 10\pi, \ldots$ Max 3 marks.

$t = \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{17\pi}{2}, \frac{21\pi}{2}, \ldots$ (1)

times 7:51am, 2:08pm, ...

7.85 14.137...

∴ Boat can re-enter harbour at 2:08pm (1) Explanation

returns to dock at 2:20pm

needs 6h50mins of fuel

$\therefore 6\frac{5}{6} \times 6 =$ 41L minimum (1)

Q26

(a) Calculate the least squares regression line for the data. 1

$E = 47.96 + 0.61t$

– Rounding needs to be consistent

(b) With evidence, describe the strength of the correlation between minutes spent on a device by a student and their mark in an examination. 1

$r = 0.45$ weak ✓ moderate ✓

* direction!

– Both weak and moderate are accepted.

– 1 mark deducted if no 'r' value.

(c) A new student is surveyed who spends 20 minutes on their device each night. Predict their score on the next examination to the nearest percentage. 1

$E = 47.96 + 0.61 \times 20$

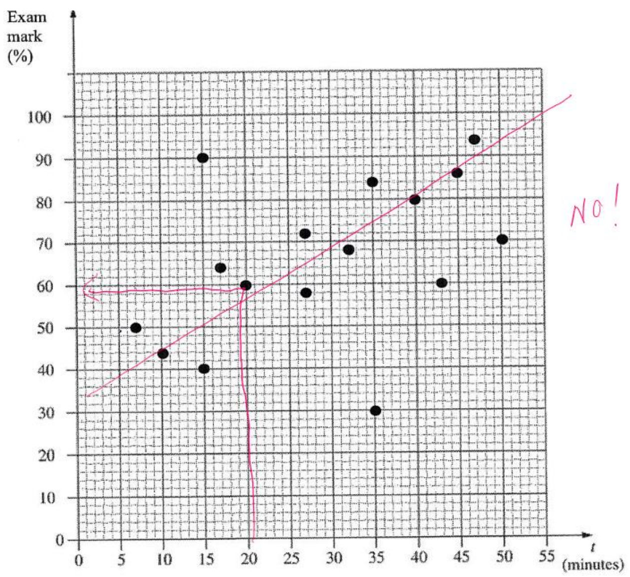
$= 60.16$

$\approx 60\%$

① Correct sub + answer

* from graph is acceptable ONLY if the regression line is drawn. – interpolated.

*no rounding penalty



(d) Another new student wanted to use the model in (a) to predict their next examination but was told that their time spent on their device was too large. What is the smallest amount of time the student could have reported spending on their device? Assume the graduation in time measurement is one minute. **2**

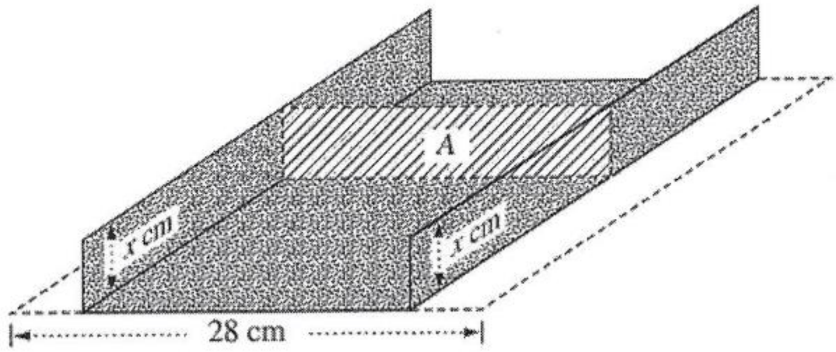
$47.96 + 0.61t > 100$ ①

$0.61t > 52.04$

$t > 85.31147\ldots$

$\therefore$ At least 86 min ①

Q27



(a) Show that the formula for the cross-sectional area, A, of the gutter is **1**

$$A = 28x - 2x^2$$

length $= 28 - x - x$

$= 28 - 2x$ ✓

$\Rightarrow A = x(28 - 2x)$

height $= x$ ✓

$A = 28x - 2x^2$

At least 2 steps

(b) Explain why this formula is only valid for $x \in (0,14)$. **1**

If $x=0$ there is no height for rectangle

if $x=14$ there is no base for rectangle.

$x > 0$ as x is length, not enough metal for $2x > 28$
$x > 14$.

$\therefore x \in (0, 14)$

Clear explanation for both $x=0$ or $x=14$.

✓ $A > 0$, use parabola

✓ $x > 0$

-1 deducted if only discussed one end of the domain.

(c) Find the maximum value of A. **3**

Well done.

For A_{max} need $\frac{dA}{dx} = 0$

i.e. $28 - 4x = 0$ (1)

$x = 7$

$\frac{d^2A}{dx^2} = -4$ (1)

$\therefore$ max at $x = 7$

must test $x=7$. Describe the nature of A at $x=7$.
→ either test the 2nd derivative OR relate to a parabola

$A_{max} = 7(28-14)$

$= 98 cm^2$ (1)

Question 28 (4 marks) **Marks**

Poorly done.

A new artist releases a song on a music streaming platform. The number of 'listens' each hour for the first 5 hours is recorded in the table below.

Hour (H)	1	2	3	4	5
Listens (L)	13	36	62	94	138

(a) Show that the predicted number of listens in the 6th hour is 206. **2**

$L_2 - L_1 = 23 = 20+3$

$L_3 - L_2 = 26 = 20+6$ AP + GP

$L_4 - L_3 = 32 = 20+12$ (1) pattern building

$L_5 - L_4 = 44 = 20+24$

$\therefore L_6 - L_5 = 20+48$

$= 68$

$L_6 = 138 + 68$

$= 206$ (1) answer

(b) How many predicted 'listens' will the song have had at the conclusion of one day? **2**

$L_{24} = 10 + 20(23) + 3(2)^{23}$

GP

$S_{24} = \frac{24}{2}(10 + 10+20(23)) + \frac{3(2^{24}-1)}{2-1}$ ① for showing correct either AP or GP

AP

$= 12(480) + 3(2^{24}-1)$

$= 50\ 337\ 405$ ② Correct answer.

Question 29 (5 marks) **Marks**

A toxic substance in a lake is degrading naturally according to the formula $A(t) = 3000m^{-0.2t}$, where A is the number of grams of the substance remaining at t years and $m > 0$ is a constant.

Some $m = 0.8$!!

(a) If $A = 2400$ after 6 years, find the value of m correct to 1 decimal place. **1**

$2400 = 3000\, m^{-1.2}$

$\frac{4}{5} = m^{-\frac{6}{5}}$

$m = \left(\frac{4}{5}\right)^{\frac{-5}{6}}$ $m = 1.2$ (1 dec pl)

(b) Fish can be reintroduced to the lake once the rate of change of the toxic substance is greater than -20. After how many years can fish be reintroduced? Answer to one decimal place. **3**

E

Need $A'(t) > -20$ ① ← poorly done, 1st derivative. ecf checked.

i.e. $(3000\ln(1.2))(-0.2\, m^{-0.2t}) > -20$

$-60\ln(\frac{6}{5})(1.2)^{-0.2t} > -2$

$1.2^{-0.2t} < \frac{1}{30\ln 1.2}$ ① correct change of base or a similar operation.

$-0.2t\ln 1.2 < -\ln(30\ln 1.2)$

$t > \frac{\ln(30\ln 1.2)}{0.2\ln 1.2}$

$t > 46.5993724\ldots$ ① Correct answer or ecf answer

$\therefore$ After $t = 46.6$ years (1 dec pl)

(c) People can swim in the lake once the amount of the toxic substance is less than 50. In how many years can this happen, to the nearest year? **1**

$3000(1.2)^{-0.2t} < 50$

$-0.2t\ln(1.2) < \ln\left(\frac{1}{60}\right)$ ∴ Safe to swim after 113 years

$t > \dfrac{\ln\left(\frac{1}{60}\right)}{-0.2\ln(1.2)}$ – must round up

Since the algebraic skills have been tested in part b

$t > 112.283611\ldots$

Question 30 (4 marks) **Marks**

A continuous random variable, X, has the following probability density function:

$$f(x) = \begin{cases} \dfrac{3\sqrt{x}}{2(\ln 2)\sqrt{x^3}} & \text{for } k \le x \le 8 \\ 0 & \text{for all other values of } x \text{ in } \mathbb{R} \end{cases}$$

(a) Find the value of k. Leave your answer as an exact value. **2**

$\int_k^8 \frac{3\sqrt{x}}{2(\ln 2)\sqrt{x^3}}\,dx = 1$ (1)

$\left[\log_2 \sqrt{x^3}\right]_k^8 = 1$

$\log_2\sqrt{8^3} - \log_2\sqrt{k^3} = \log_2 2$

$\sqrt{\left(\frac{8}{k}\right)^3} = 2$

$\frac{8}{k} = \sqrt[3]{4}$

Variations:

$\sqrt[3]{2^7}$, $\frac{8}{2^{2/3}}$, $e^{\frac{\ln 128}{3}}$, ≈ 5.03968 $k = \frac{8}{\sqrt[3]{4}}$

$\frac{8}{e^{\frac{2\ln 2}{3}}}$ $k = 4\sqrt[3]{2}$

(b) Find the median value of X as an exact value. **2**

$$\int_{4\sqrt[3]{2}}^{m} \frac{3\sqrt{x}}{(2\ln 2)\sqrt{x^3}}\,dx = 0.5$$ (1)

$$\log_2 \sqrt{m^3} - \log_2 \sqrt{(4\sqrt[3]{2})^3} = \frac{1}{2}$$

$$\log_2 m^3 - \log_2 2\times 4^3 = \log_2 2$$

$$\frac{m^3}{2\times 4^3} = 2$$

$m = 4\sqrt[3]{4}$ is the median value

$$\sqrt[3]{2^8} = 2^{\frac{8}{3}} = \sqrt[3]{128} \times 4^{\frac{1}{6}} \quad \frac{e^{\ln 8}}{e^{\frac{\ln 4}{6}}} = \frac{2}{e^{-\frac{1}{6}}}$$

$m \approx 6.3496$